

Exact Solutions to the Generalised Lienard Equation

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Abstract

Many new solitary wave solutions of the recently studied Lienard equation are obtained by mapping it to the field equation of the ϕ^6 -field theory. Further, it is shown that the exact solutions of the Lienard equation are also the exact solutions of the various perturbed soliton equations. Besides, we also consider a one parameter family of generalised Lienard equations and obtain exact solitary wave solutions of these equations and show that these are also the exact solutions of the various other generalised nonlinear equations.

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I. Introduction

Recently Kong [1] has obtained an exact solitary wave solution of the Lienard equation. The study of the Lienerd equation is very useful because finding solitary wave solutions of some important equations such as RR equation [2], the A-equation [3] and the GI equation [4] which are proposed as integrable conditions for the nonlinear Schrödinger equation and also host of other equations [1] can be reduced to finding the solutions of the Lienard equation. Moreover, as we will show in this letter, exact solutions of various perturbed soliton equations like cubic nonlinear Schrödinger equation, KdV equation as well as the recently considered Wadati-Segur-Ablowitz equation [5] can also be obtained by reducing these perturbed equations to the Lienard form.

With these applications in mind, in this letter we report various new topological and periodic solitary wave solutions of the Lienard equation. These new solutions are obtained by identifying the Lienard equation with the equation of motion of the well known nonlinear ϕ^6 field theory in (1+1) dimensions. The analysis of the ϕ^6 field theory has been done in great detail in [6] and this identification results in several new exact solutions of the Lienard equation. In particular, it turns out that the most general solution of the Lienard equation can be expressed in terms of Weirstrass function. Besides, we also obtain several exact solutions of a one parameter family of the generalised Lienard equations and show that the exact solutions of the other generalised equations like the generalised RR-equation, the generalised A-equation, the generalised GI-equation and the host of other generalised nonlinear equations can be expressed in terms of the exact solutions of this equation.

II. New solitary wave solutions of the Lienard equation

The Lienard equation is given by [1]

$$\phi''(\zeta) + l\phi(\zeta) + m\phi^3(\zeta) + n\phi^5(\zeta) = 0 \quad (1)$$

where l,m and n are constant coefficients. Kong has obtained a nontopological solitary wave solution of this equation of the form

$$\phi(\zeta) = \frac{1}{[A + B \cosh D(\zeta + \zeta_0)]^{1/2}} \quad (2)$$

in case either $l < 0, n > 0$ or $l, n < 0, m > 0$ and $nl/m^2 < 3/16$.

Let us first show that the Lienard equation can be identified with the field equations of the ϕ^6 nonlinear field theory. which in (1+1) dimensions is used as a model for structural phase transition [6]. The onsite potential corresponding to this model is given by

$$V(\phi_i) = B\phi_i^2 + A\phi_i^4 + C\phi_i^6 \quad (3)$$

In the continuum limit, the corresponding equation of motion is given by [6]

$$m \frac{\partial^2 \phi}{\partial t^2} - mC_0^2 \frac{\partial^2 \phi}{\partial x^2} + 2B\phi + 4A\phi^3 + 6C\phi^5 = 0 \quad (4)$$

In the travelling wave frame $\zeta = x - vt$, $\phi(x, t) = \phi(\zeta)$ and this equation reduces to

$$\frac{d^2 \phi}{dS^2} - \phi - \frac{2A}{B}\phi^3 - \frac{3C}{B}\phi^5 = 0 \quad (5)$$

where $S = \zeta/\zeta_0$ and $\zeta_0^2 = m(C_0^2 - v^2)/2B$. Thus this ϕ^6 -field equation is nothing but the Lienard eq. (1) considered by Kong [1], for the choice of the coefficients $l = -1, m = -2A/B$ and $n = -3C/B$. Accordingly, all the solutions of eq. (5) will also be the solution of the Lienard equation (1) for the suitable values of the coefficients l, m and n .

The onsite-potential $V(\phi_i)$ as given by eq. (3) whose average value essentially corresponds to the static free energy of the system exhibits a first order phase transition [6] if $B, C > 0, A < 0$ and $0 < a < 3/2$ where $a = \frac{9BC}{2|A|^2}$. Under these conditions the potential has three minima. On the other hand if $B < 0, C > 0$ then the potential has two degenerate minima irrespective of the sign of A. Finally, in case $C < 0$, then the

Lienard equation can be identified with the equation of motion of a classical particle (of unit mass) moving in an anharmonic potential [7].

Various solutions of eq. (5) have been presented in [6] and [7]. In particular, it has been shown that the exact solution of eq. (5) can be expressed in terms of the Weirstrass function. In view of the above identification, we thus see that the most general solution of the Lienard eq. (1) can be written in terms of Weirstrass function. On borrowing the results of [6,7] some simple special solutions of the Lienard equation (1) are

- (i) the topological solitary wave (kink or domain wall) solutions which are given by

$$\phi(\zeta) = \pm \frac{A \tanh(Bx)}{[1 - C \tanh^2(Bx)]^{1/2}} \quad (6)$$

where $A = [-2l(1 - 3C)(1 - C)/m(2 - 3C)]^{1/2}$, $B = [l/(2 - 3C)]^{1/2}$ and $ln/m^2 = -3C(2 - 3C)/[2(1 - 3C)]^2$. Here C is an arbitrary constant with $C < 1$ so that the solution is nonsingular while $x = (\zeta + \zeta_0)$. As x goes from $-\infty$ to $+\infty$, $\phi(\zeta)$ goes from one absolute minima of the potential to the other. Notice that (i) if $2/3 < F < 1$ then $n, l < 0, m > 0$ (ii) if $1/3 < C < 2/3$ then $n < 0, l, m > 0$ (iii) if $0 < C < 1/3$ then $n, m < 0, l > 0$ (iv) if $C < 0$ then $n, l > 0, m < 0$.

- (ii) For $n, l > 0, m < 0$ and $nl/m^2 = 1/4$ there is a special solution given by

$$\phi(\zeta) = \pm \frac{(-2l/m)^{1/2}x}{[(3/l) + x^2]^{1/2}}. \quad (7)$$

Apart from these, there are few other exact solutions of the Lienard eq. (1) which we shall mention below in Sec.4 when we discuss the various solutions of the generalised Lienard equations. As a corollary, we can say that the RR-equation, the A-equation and the GI-equation and the host of other equations [1] which can be reduced to the Lienard eq. (1) will also have the new exact solitary wave solutions as given above as well as in Sec.4

III. Applications: Perturbed Soliton Equations

Perturbed soliton equations represent those which differ slightly from the standard soliton equations and represent physically the more realistic experimental situations, especially the effect of various forms of dissipation, dispersion etc. which are treated as perturbation. Various perturbative methods have been developed to study these perturbed soliton equations [8]. In this section we shall show that some of the well known perturbed soliton equations can be reduced to the Lienard form (1) by using suitable ansatz. Thus the exact solutions of the Lienard eq. (1) also represent the exact solutions of these perturbed equations.

(i) The Perturbed Cubic Nonlinear Schrödinger Equation:

This equation is represented by [8]

$$i\phi_t + \phi_{xx} + 2|\phi|^2\phi = i \in R(\phi) \quad (8)$$

where $\epsilon \ll 1$ and $R(\phi)$ is some specified function of the solution $\phi(x, t)$ representing the perturbation. For example, $\epsilon R(\phi) = \gamma\phi$ provides a simple description of dissipative process and $\epsilon R(\phi) = \gamma\phi_{xx}$ induces diffusion effects on the system. Here we consider $R(\phi) = |\phi|^2\phi_x$. To get the exact solutions of the perturbed eq. (8) we use the ansatz

$$\phi(x, t) = a(\zeta)\exp(i[\psi(\zeta) - wt]) \quad (9)$$

where $\zeta = x - vt$. Substituting eq. (9) in eq. (8) we get

$$-va' + 2a'\psi' + a\psi'' - \epsilon a^2a' = 0 \quad (10)$$

and

$$va\psi' + wa + a'' - a\psi'^2 + 2a^3 + \epsilon a^3\psi' = 0 \quad (11)$$

Following Kong [1], let $\psi'(\zeta) = E + Da^2(\zeta)$ where E and D are constants. Substituting this in eq. (10) we get $E = v/2$ and $D = \epsilon/4$. Similarly substitution of $\psi'(\zeta)$ in eq. (11)

gives

$$a'' + \left(w + \frac{v^2}{4}\right)a + \left(2 + \frac{\epsilon v}{2}\right)a^3 + \frac{3}{16}\epsilon^2 a^5 = 0 \quad (12)$$

which is nothing but the Lienard eq. (1) for the choice of the coefficients $l = (w + \frac{v^2}{4})$, $m = (2 + \frac{\epsilon v}{2})$ and $n = \frac{3}{16}$. Thus, for the appropriate choice of the constant coefficients, various exact solutions of the perturbed cubic nonlinear Schrödinger eq. (8) are immediately obtained in terms of the exact solutions of the Lienard eq. (1) as given in Secs. 2 and 4 (below).

(ii) Perturbed Modified KdV Equation:

Similarly consider the perturbed modified KdV equation

$$\phi_t + \alpha\phi^2\phi_x + \gamma\phi_{xxx} = \epsilon\phi^4\phi_x \quad (13)$$

where the effect of higher order nonlinear term is considered as a perturbation to the well known integrable modified KdV equation $\phi_t + \alpha\phi^2\phi_x + \gamma\phi_{xxx} = 0$. Using the transformation $\zeta = x - vt$, eq. (13) can be written as

$$-\nu\phi_\zeta + \alpha\phi^2\phi_\zeta + \gamma\phi_{\zeta\zeta\zeta} = \epsilon\phi^4\phi_\zeta \quad (14)$$

Integrating w.r.t ζ we get

$$\gamma\phi_{\zeta\zeta} - \nu\phi + \frac{\alpha}{3}\phi^3 - \frac{\epsilon}{5}\phi^5 = 0 \quad (15)$$

which is of the Lienard form (1) with the identification of the coefficients $l = -\frac{\nu}{\gamma}$, $m = \frac{\alpha}{3\gamma}$ and $n = -\frac{\epsilon}{5\gamma}$. Hence various exact solutions of the perturbed modified KdV eq. (13) are also known in terms of the exact solutions of the Lienard equation as given in Secs. 2 and 4 (below).

(iii) Perturbed Wadati-Segur-Ablowitz (WSA) equation:

Recently Wadati et al. [5] have introduced the equation

$$iu_x + u_{tt} + 2\sigma |u^2| u - \epsilon u_{xt} = 0 \quad (16)$$

to study certain instabilities of the modulated wave trains and obtained an exact solitary wave solution of this equation. Later Kong and Zhang [9] obtained another exact solitary wave solution of this equation. Here, we point out that if the WSA equation is perturbed by a term of the form $| u |^2 u_x$, then the resulting perturbed WSA equation can be reduced to the Lienard form by using the ansatz for $u(x,t)$ as given in eq. (9) and following the procedure as used for the perturbed cubic nonlinear schrödinger equation.

Yet another form of the perturbed WSA equation which can be casted in the Lienard form is given by

$$iu_x + u_{tt} + 2\sigma | u |^2 u - \alpha u_{xt} = \epsilon | u |^4 u \quad (17)$$

where $\epsilon \ll 1$ and the higher order nonlinear term is considered as a perturbation to the WSA eq. (16). Using an ansatz [10] which is different from that in eq. (9) i.e.

$$u(x,t) = e^{i\eta} \phi(\rho) \quad (18)$$

where $\rho = ax - vt$ and $\eta = Kx - \omega t$ and substituting in eq. (17) one again obtains a Lienard equation.

We would also like to point out that in the WAS eq. (16) if we replace the $| u |^2 u$ term by $i | u |^2 u_x$ term, then the resulting equation, which we term as modified WSA equation, can also be reduced to the Lienard form by using the ansatz for $u(x,t)$ as in eq. (9).

IV. The Generalised Lienard Equations and their Solutions

(1) The Generalised Lienard Equations:

Inspired by the success of the Lienard eq. (1), we consider a one parameter family of generalised Lienard equations with p 'th order nonlinearity i.e. consider

$$\phi''(\zeta) + l\phi(\zeta) + m\phi^{p+1}(\zeta) + n\phi^{2p+1}(\zeta) = 0 \quad (19)$$

where l, m and n are constant coefficients and $p = 1, 2, 3, \dots$. The Lienard equation in [1] corresponds to the $p=2$ case of these generalised equations. As compared to the onsite potential for the Lienard equation as given by eq. (3), the generalised Lienard equation above corresponds to an onsite potential

$$V(\phi_i) = B\phi_i^2 + A\phi_i^{p+2} + C\phi_i^{2p+2}, \quad (20)$$

Some exact solutions of the generalised Lienard eqs. (19) and their applications have been reported in [11]. We now present four exact solutions of eqs. (19).

(a) The exact topological solutions of eq. (19) can be written as [11]

$$\phi^p(\zeta) = \phi_0[1 \pm \tanh Bx], \quad (21)$$

in case $l, n < 0$ and $ln/m^2 = (p+1)/(p+2)^2$. Here $\phi_0 = -l(p+2)/2m$, $B = p\sqrt{-l}/2$ while x is as defined in Sec. 2. Notice that if p is even then $m > 0$ while if p is odd then it could be of either sign. For the special case of $p = 2$ (Lienard equation) we then obtain the well known [12] topological solution. Notice that for any even (odd) p , the topological solution has been obtained at the point where the potential (eq. (20)) has three (two) degenerate minima.

(b) Another exact solution of eq. (19) is given by

$$\phi(\zeta) = \frac{1}{[A + B \cosh Dx]^{1/p}}, \quad (22)$$

where $A = -m/[l(p+2)]$, $B = [\frac{m^2}{l^2(p+2)^2} - \frac{n}{l(p+1)}]^{1/2}$ and $D = p\sqrt{-l}$. Note that this solution is acceptable if either $l, n < 0, m > 0$ and $ln/m^2 < (p+1)/(p+2)^2$ or $l < 0, n > 0$ and m arbitrary. For $p=2$, this reduces to the solution obtained by Kong [1]. For odd p , solution with B replaced by $-B$ is also allowed in either of the two cases. Further, in both cases m could be either positive or negative.

(c) Yet another exact solution of eq. (19) is the periodic solution given by

$$\phi(\zeta) = \frac{1}{[A \pm B \sin D\zeta]^{1/p}}, \quad (23)$$

where A and B are as given above (below eq. (22)) while $D = p\sqrt{l}$. This is an acceptable solution if $l, n > 0, m < 0$ and $nl/m^2 < (p+1)/(p+2)^2$. Note that for odd p, m could have either sign. For $p = 2$ it gives us the periodic solution of the Lienard eq. (1). It may however be noted that as shown in [6,7], the most general periodic solution to the Lienard eq. (1) is infact the Jacobi elliptic function.

(d) For the special case of $l = 0, n > 0, m < 0$ there is an exact solution of eq. (19) given by

$$\phi(\zeta) = \frac{1}{[A + Bx^2]^{1/p}} \quad (24)$$

where $A = -n(p+2)/[2m(p+1)]$ and $B = -mp^2/[2(p+2)]$. Note again that for p odd, m can either be positive or negative.

(2) The Generalised RR-equation:

We consider the following generalised RR-equation

$$\phi_{xt} - \beta_1 \phi_{xx} + \phi + iT\beta_2 |\phi|^p \phi_x = 0, \quad p = 1, 2, 3\dots \quad (25)$$

To get the exact solution of this equation we use the ansatz [1] as given by eq. (9) and obtain

$$\phi_t = [-iva'\psi' - iwa - va'] \exp[i(\psi(\zeta) - wt)] \quad (26)$$

Similarly, we can obtain ϕ_x , ϕ_{xt} and ϕ_{xx} which are also given in [1]. Substituting these values in eq. (25) we get

$$-(v + \beta_1)[a(\zeta)\psi''(\zeta) + 2a'(\zeta)\psi'(\zeta)] - wa'(\zeta) + T\beta_2 a^p(\zeta)a'(\zeta) = 0, \quad (27)$$

and

$$-(v + \beta_1)[a''(\zeta) - a(\zeta)\psi'^2(\zeta)] + wa(\zeta)\psi'(\zeta) - T\beta_2 a^{p+1}(\zeta)\psi'(\zeta) + a(\zeta) = 0 \quad (28)$$

Let

$$\psi'(\zeta) = E + Da^p(\zeta) \quad (29)$$

where E and D are constants. Substituting this in eq. (27) we get

$$E = -\frac{w}{2(v + \beta_1)} \quad \text{and} \quad D = \frac{T\beta_2}{(p+2)(v + \beta_1)} \quad (30)$$

From eq. (28) we then obtain

$$a''(\zeta) - \frac{[4(v + \beta_1) - w^2]}{4(v + \beta_1)^2} a(\zeta) - \frac{T\beta_2 w}{2(v + \beta_1)^2} a^{p+1}(\zeta) + \frac{(p+1)\beta_2^2}{(p+2)^2(v + \beta_1)^2} a^{2p+1}(\zeta) = 0 \quad (31)$$

This is of the form of the generalised Lienard eq. (19) and thus the various exact solutions of the generalised RR-equation can be expressed in terms of the exact solutions of the generalised Lienard eq. 919) as mentioned above. Note that for p=2 case, the generalised RR eq. (25) reduces to the well known RR equation [2] (see eq. (21) of [1]).

Proceeding in the same way, one can show that the generalised A-equations [3]

$$i\phi_t = \phi_{xx} - 4i |\phi|^{p-2} \phi^2 \bar{\phi}_x + 8 |\phi|^{2p} \phi \quad (32)$$

as well as the generalised GI-equations [4] as given by

$$i\phi_t + \phi_{xx} + 2i\delta |\phi|^{p-2} \phi^2 \bar{\phi}_x + \beta |\phi|^p \phi + 2\delta^2 |\phi|^{2p} \phi = 0 \quad (33)$$

can be reduced to the generalised Lienard eq. (19).

Similarly, there are other generalised equations whose exact solutions can also be obtained by reducing them to the generalised Lienard eq. (19). Some of these are [1]

$$(i) i\phi_t + \phi_{xx} \pm i(|\phi|^p \phi)_x = 0$$

$$(ii) i\phi_t + \phi_{xx} \pm i\alpha(|\phi|^p \phi)_x \mp \beta |\phi|^p \phi = 0$$

- $$(iii) i\phi_t + \phi_{xx} \pm i\delta |\phi|^p \phi_x = 0$$
- $$(iv) i\phi_t + \phi_{xx} + i\alpha(\phi\bar{\phi}_x + \bar{\phi}\phi_x) |\phi|^{p-2} \phi = 0$$
- $$(v) i\phi_t + (\phi_{xx} \pm \phi_{yy}) + \beta |\phi|^p \phi + \delta |\phi|^{2p} \phi - i\delta \text{div}(|\phi|^p \phi) = 0$$

V. Conclusion

In conclusion, we have obtained several new solitary wave solutions of the Lienard equation recently considered by Kong [1]. An important observation we have made here is the identification of the Lienard equation with the ϕ^6 field theory in (1+1) dimension which describes structural phase transition in various physical systems [6]. This identification has enabled us to find many new solutions of the Lienard eq. (1). As an application, we have shown how the exact solutions of the various perturbed soliton equations can also be obtained by reducing these equations to Lienard form by using some appropriate ansatz.

Very recently, soliton solutions of the complex Ginzberg-Landau equation have been obtained[13]. We wish to point out that the eq. (3) of their paper [13] is immediately obtained from Lienard eq. (1) by multiplying it by ϕ , integrating it and choosing the constant of integration to be zero. Thus the four exact solutions of the Lienard eq. (1) as given by eqs. (21) to (24) (with $p = 2$) are also the exact solutions of the complex Ginzburg-Landau eq. (3) of ref.[13].

Finally, in view of the interest to study the generalised soliton equations [10], we have considered a one parameter family of the generalised Lienard eqs. (19) and obtained its several exact solutions. As an application, we have also studied the generalised RR equation, A-equation, GI-equation and host of other generalised equations and shown that all of them can be reduced to the generalised Lienard eqs. (19) so that the exact solutions of these generalised equations can also be immediately obtained.

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